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Extremality and the Global Markov Property II: The Global Markov Property for Non-FKG Maximal Gibbs Measures

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We give a condition on a Gibbs measure for an attractive Markov specification, which assures extremality and the global Markov property. As an example of application we consider the class of attractive Markov specifications defined on a compact configuration space over a two-dimensional lattice by the interaction Hamiltonians (assumed to have a finite set of periodic ground configurations) satisfying Peierl's condition. We prove that each extremal Gibbs measure for such a specification, at sufficiently low temperature, has the global Markov property.

KEY WORDS: Lattice spin systems with attractive force; multiphase region; Gibbs measures; extremality; Markov property.

INTRODUCTION

In the author's paper,[12] the criteria on the Gibbs measures for extremality and the global Markov property have been given in the case of attractive Markov local specifications on a lattice with arbitrary single-spin state space. Using one of these criteria, the global Markov property has been proven for FKG-maximal Gibbs measures, which define the Euclidean field theories on a lattice.

In the present paper (on the basis of the ideas of[12]) we formulate another natural criterion for (extremality and) the global Markov property, which gives the possibility of proving the global Markov

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property also for non-FKG-maximal Gibbs measures (for a Markov attractive specification). We apply this criterion to the systems with attractive short range interactions which fulfil Peierl's condition[3,5,7,8,10] on a two-dimensional lattice. We show that in sufficiently low temperature the non-FKG-maximal Gibbs measures (which exist if temperature is sufficiently close to zero) also have the global Markov property. It is expected that the method of proof used in the example can be applicable in the important case of Euclidean fields on a lattice at low temperature regions, when the phase transitions can occur, since the theory of the works[3,5,7,8,10] for Euclidean fields exists (see Imbrie, Ref. 25 in Ref. 11).

The organization of the paper is as follows: In Section 1 we give the abstract formulation of our criterion for any attractive and Markov specification on a lattice with arbitrary single-spin state space. In Sections 2 and 3 we apply the example to the class of systems of finite spins on a two-dimensional lattice.

1. Let $\mathbb{L}$ be a countable set, $\mathscr{L}$ the family of its finite subsets, and $\mathscr{L}_o$ a countable base of $\mathscr{L}$, i.e.,

$$\mathscr{L}_o := \left\{ \Lambda_n \in \mathscr{L}, n \in \mathbb{N}: \Lambda_n \subset \Lambda_{n+1}, \bigcup_n \Lambda_n = \mathbb{L} \right\} \tag{1}$$

For $\Lambda \subset \mathbb{L}$ we write $\Lambda^c \equiv \mathbb{L} \backslash \Lambda$. To each point $i \in \mathbb{L}$ we associate a set $\partial\{i\}^c \subset \mathbb{L}$ of the nearest neighbors of i. We define the boundary $\partial\Lambda$ of a set $\Lambda \subset \mathbb{L}$ by

$$\partial\Lambda := (i \in \Lambda: \Lambda^c \cap \partial\{i\}^c \neq \varnothing) \tag{2}$$

For any $i \in \mathbb{L}$, let $(Y_i, \mathscr{Y}_i)$ be a Borel set $Y_i \subset \mathbb{R}$ with the σ algebra $\mathscr{Y}_i$ of Borel subsets of Y_i.

Let $(\Omega, \Sigma) := \mathsf{X}_{i \in \mathbb{L}} (Y_i, \mathscr{Y}_i)$. For $Q \subset \mathbb{L}$ by Σ_Q we denote the σ algebra generated by the sets of the form $\mathsf{X}_{i \in \Lambda} A_i \times \mathsf{X}_{i \in \Lambda^c} Y_i$, where $\Lambda \in \mathscr{L}$, $\Lambda \subseteq Q$, and $A_i \in \mathscr{Y}_i$. By definition, $\Sigma = \Sigma_{\mathrm{L}}$. The σ algebra at infinity is defined by

$$\Sigma_\infty := \bigcap_{\Lambda \in \mathscr{L}} \Sigma_{\Lambda^c} \tag{3}$$

For $Q \subset \mathbb{L}$ and $\omega \in \Omega$, the restriction of ω to Q is $\omega_Q := \{\omega_i: i \in Q\}$.

On (Ω, Σ) we have natural measurable, directed upward and downward order $\preccurlyeq$ defined as follows

$$\omega, \omega' \in \Omega: \omega \preccurlyeq \omega' \Leftrightarrow \forall i \in \mathbb{L} \qquad \omega_i \leqslant \omega'_i \tag{4}$$

We define

$$(\omega \wedge \omega')_i := \min(\omega_i, \omega'_i); \qquad (\omega \vee \omega')_i := \max(\omega_i, \omega'_i) \tag{5}$$

We say that a function $F: \Omega \to \mathbb{R}$ is increasing if

$$\omega \preccurlyeq \omega \Rightarrow F(\omega) \leqslant F(\omega') \tag{6}$$

For $\Lambda \subset \mathbb{L}$, $\mathscr{A}_\Lambda$ (respectively, $\mathscr{A}_\Lambda^\uparrow$) denotes the set of bounded Σ_Λ measurable (and increasing, respectively) functions. If $\Lambda = \mathbb{L}$ the superscript Λ will be omitted. We distinguish the subsets of nonnegative functions by adding $+$ as a superscript to the sets of functions considered (e.g., $\mathscr{A}_\Lambda^+, \mathscr{A}_\Lambda^{\uparrow +}$).

By $\mathscr{M}$ we denote the set of probability measures on (Ω, Σ). For $\mu \in \mathscr{M}$ and a measurable function F, by $\mu(F)$ or simply μF we denote the expectation value of F with a measure μ. The conditional expectation of F with respect to a σ algebra $\Sigma' \subset \Sigma$, associated to a measure $\mu \in \mathscr{M}$, is denoted by $E_\mu(F \mid \Sigma')$. For $F, G \in L_2(\mu)$ we write

$$\mu(F, G) := \mu(F \cdot G) - \mu(F)\, \mu(G) \tag{7}$$

From our assumptions about the order $\preccurlyeq$ it follows (see, e.g., [9]) that $\bigcup_{\mathscr{L}} \mathscr{A}_\Lambda^\uparrow$ (and so $\bigcup_{\mathscr{L}} \mathscr{A}_\Lambda^{\uparrow +}$) is a determining class for $\mathscr{M}$, i.e., for any $\mu, \mu' \in \mathscr{M}$ and $\Lambda \in \mathscr{L}$

$$\forall F \in \mathscr{A}_\Lambda^\uparrow \quad \mu(F) = \mu'(F) \Rightarrow \mu\mid_{\Sigma_\Lambda} = \mu'\mid_{\Sigma_\Lambda} \tag{8}$$

Hence, two measures given as the limits of sequences of measures indexed by $\Lambda \in \mathscr{L}_o$: $\mu := \lim_{\mathscr{L}_o} \mu_\Lambda$ and $\mu' := \lim_{\mathscr{L}_o} \mu'_\Lambda$ are equal if for any $F \in \bigcup_{\mathscr{L}} A_\Lambda^\uparrow$ (or $\bigcup_{\mathscr{L}} A_\Lambda^{\uparrow +}$) we have

$$\lim_{\mathscr{L}_o} |\mu_\Lambda F - \mu'_\Lambda F| = 0 \tag{8'}$$

In the set $\mathscr{M}$ we define the FKG order by

$$\mu \leqslant_{\mathrm{FKG}} \mu' \Leftrightarrow \forall F \in \mathscr{A}^\uparrow \qquad \mu(F) \leqslant \mu'(F) \tag{9}$$

For $\mu \in \mathscr{M}$, if

$$\forall \Lambda \in \mathscr{L} \qquad \forall F \in \mathscr{A}_\Lambda \qquad E_\mu(F \mid \Sigma_{\Lambda^c}) \in \mathscr{A}_{\partial \Lambda^c} \tag{10}$$

we say that μ has the local Markov property, and if

$$\forall Q \subset \mathbb{L} \qquad \forall F \in \mathscr{A}_Q \qquad E_\mu(F \mid \Sigma_{Q^c}) \in \mathscr{A}_{\partial Q^c} \tag{11}$$

we say that μ has the global Markov property and we write $\mu \in \mathrm{GMP}$. The global Markov property implies the local one, but the converse is not true in general (see, e.g., Ref. 4 and 11).

A local specification[2,9] is a family $\mathscr{E} := \{E_{\Lambda^c}\}_{\Lambda \in \mathscr{L}}$, which consists of functions

$$E_{\Lambda^c}: \Omega \times \Sigma \to [0, 1] \tag{12}$$

such that

(i) $\forall \Lambda \in \mathscr{L}$, $\forall \omega \in \Omega$, $E^{\omega}_{\Lambda^c}(\cdot) \in \mathscr{M}$ and the restriction of this measure to Σ_{Λ^c} coincides with the point measure δ_ω

(ii) $\forall \Lambda \in \mathscr{L}$ $\forall F \in \mathscr{A}$ $E^{\cdot}_{\Lambda^c}(F) \in \mathscr{A}_{\Lambda^c}$

(iii) The compatibility condition

$$\Lambda_1, \Lambda_2 \in \mathscr{L}: \Lambda_1 \subset \Lambda_2 \Rightarrow E_{\Lambda_2^c} E_{\Lambda_1^c} = E_{\Lambda_2^c} \tag{13}$$

A local specification $\mathscr{E} = \{E_{\Lambda^c}\}_{\Lambda \in \mathscr{L}}$ is called attractive, if

$$\forall \Lambda \in \mathscr{L} \qquad \forall F \in \mathscr{A}^{\uparrow} \qquad E_{\Lambda^c}(F) \in \mathscr{A}^{\uparrow} \tag{14}$$

or, equivalently

$$\forall \omega, \omega' \in \Omega \qquad \omega \preccurlyeq \omega' \Rightarrow E^{\omega}_{\Lambda^c} \leqslant_{\mathrm{FKG}} E^{\omega'}_{\Lambda^c} \tag{14'}$$

Markov, if

$$\forall \Lambda \in \mathscr{L} \qquad \forall F \in \mathscr{A}_{\Lambda} \qquad E^{\cdot}_{\Lambda^c}(F) \in \mathscr{A}_{\partial \Lambda^c} \tag{15}$$

The set of Gibbs measures for $\mathscr{E}$ is defined by

$$G(\mathscr{E}) := \{\mu \in \mathscr{M}: \forall \Lambda \in \mathscr{L} \ \mu E_{\Lambda^c} = \mu\} \tag{16}$$

The set of its extremal points (i.e., the set of these Gibbs measures for $\mathscr{E}$, which cannot be represented as a convex linear combination of other elements from $G(\mathscr{E})$) is denoted by $\partial G(\mathscr{E})$.

If $\mathscr{E}$ is attractive, then for μ, $\tilde{\mu} \in G(\mathscr{E})$ such that

$$\mu = \lim_{\mathscr{L}_o} \mu' E_{\Lambda^c} \qquad \text{and} \qquad \tilde{\mu} = \lim_{\mathscr{L}_o} \mu'' E_{\Lambda^c} \tag{17}$$

for some μ', $\mu'' \in \mathscr{M}$, $\mu' \leqslant_{\mathrm{FKG}} \mu''$ we have

$$\mu \leqslant_{\mathrm{FKG}} \tilde{\mu} \tag{18}$$

In the special case $\mu' = \delta_\omega$, $\mu'' = \delta_{\tilde{\omega}}$ with ω, $\tilde{\omega} \in \Omega$, $\omega \preccurlyeq \tilde{\omega}$ we have

$$E^{\omega}_{\Lambda^c} \leqslant_{\mathrm{FKG}} E^{\tilde{\omega}}_{\Lambda^c} \qquad \text{and so} \qquad \lim_{\mathscr{L}_o} E^{\omega}_{\Lambda^c} \leqslant_{\mathrm{FKG}} \lim E^{\tilde{\omega}}_{\Lambda^c} \tag{19}$$

If, moreover, $\mathscr{E}$ is compact on (Ω, Σ) in the sense (see, e.g., Ref. 1) that for each $\omega \in \Omega$, $\{E^{\omega}_{\Lambda^c}\}_{\Lambda \in \mathscr{L}}$ is compact and $G(\mathscr{E})$ is compact (in the weak topology in $\mathscr{M}$), or more generally, $\mathscr{E}$ is compact with respect to some $\tilde{\mathscr{M}} \subset \mathscr{M}$ in the sense that for each $\mu \in \tilde{\mathscr{M}}$, $\{\mu E_{\Lambda^c}\}_{\Lambda \in \mathscr{L}}$ is compact and $G(\mathscr{E}) \cap \tilde{\mathscr{M}}$ is compact (in the relative topology in $\tilde{\mathscr{M}}$), then, under the additional assumption in the second case, that

$$\begin{gathered} \forall \mu', \mu'' \in \tilde{\mathscr{M}} \qquad \exists \mu' \wedge \mu'', \mu' \vee \mu'' \in \mathscr{M} \\ \mu' \wedge \mu'' \leqslant_{\mathrm{FKG}} \mu', \mu'' \leqslant_{\mathrm{FKG}} \mu' \vee \mu'' \end{gathered} \tag{20}$$

there exist unique FKG-maximal measures μ^+, $\mu^- \in G(\mathscr{E})$, i.e.

$$\forall \mu \in G(\mathscr{E}) \text{ (respectively, } \forall \mu \in G(\mathscr{E}) \cap \tilde{\mathscr{M}}) \qquad \mu_- \leqslant_{\mathrm{FKG}} \mu \leqslant_{\mathrm{FKG}} \mu_+ \tag{21}$$

(Let us note that if $\exists \mu_+$, $\mu_- \in G(\mathscr{E})$ FKG-maximal then $G(\mathscr{E})$ is compact.)

If $\mathscr{E}$ is Markov then, from definitions (10) and (15), any $\mu \in G(\mathscr{E})$ has the local Markov property, but it can happen that $\mu \notin \mathrm{GMP}$.[4,11] It is known (see, e.g., Ref. 12) that for $\mathscr{E}$ compact attractive and Markov we have for FKG-maximal measures μ_+, $\mu_- \in G(\mathscr{E})$ that

$$\mu_+, \mu_- \in \mathrm{GMP} \tag{22}$$

We now formulate a condition which gives $\mu \in G(\mathscr{E}) \cap \mathrm{GMP}$ for $\mathscr{E}$ compact, attractive, and Markov also for non-FKG-maximal Gibbs measures.

In Section 3 we give the examples for our results, namely, that there are Gibbs measures extremal and $\neq \mu_\pm$ which have the global Markov property.

With the above notations and definitions we have

Proposition 1. Let $\mathscr{E}$ be an attractive specification on a standard Borel space (Ω, Σ) with a partial (measurable, directed upward and downward) order $\preccurlyeq$. Let $\mu \in G(\mathscr{E})$.

If there exists $\omega^o \in \Omega$, such that

$$\lim_{\mathscr{L}_o} \mu E^{\omega \vee \omega^o}_{\Lambda^c} = \lim_{\mathscr{L}_o} \mu E^{\omega \wedge \omega^o}_{\Lambda^c} \tag{23}$$

(where ω is the integration variable) then

$$\mu \in \partial G(\mathscr{E})$$

and if $\mathscr{E}$ is Markov then

$$\mu \in \mathrm{GMP} \qquad \square$$

Remarks. In the proof we do not use any other special features of an order $\preccurlyeq$ besides those mentioned in the formulation of Proposition 1. We also do not use explicitly $\mathbb{L}$ as a lattice. What we need only is the possibility of restriction to a subset of $\mathbb{L}$ and "patching together" two configurations from Ω. Note also that the compactness is not used in above proposition. The condition (23) is the condition on a Gibbs measure.

This proposition is the corollary from Ref. 12, Proposition 2; however, for the reader's convenience, we write the full proof in the case under consideration.

Proof. (Extremality) For any $F \in \mathscr{A}^{\uparrow}$ and for each $\omega \in \Omega$ we have (from attractivity of $\mathscr{E}$)

$$E_{\Lambda^c}^{\omega \wedge \omega^o}(F) \leqslant E_{\Lambda^c}^{\omega^o}(F) \leqslant E_{\Lambda^c}^{\omega \vee \omega^o} \tag{24}$$

and

$$E_{\Lambda^c}^{\omega \wedge \omega^o}(F) \leqslant E_{\Lambda^c}^{\omega}(F) \leqslant E_{\Lambda^c}^{\omega \vee \omega^o}(F) \tag{25}$$

Hence

$$\lim_{\mathscr{L}_o} \mu \, |E_{\Lambda^c}^{\omega}(F) - E_{\Lambda^c}^{\omega^o}(F)| \leqslant \lim_{\mathscr{L}_o} \mu[E_{\Lambda^c}^{\omega \vee \omega^o}(F) - E_{\Lambda^c}^{\omega \wedge \omega^o}(F)] = 0 \tag{26}$$

so there is a subsequence $\mathscr{L}'_o \subset \mathscr{L}_o$ such that

$$\lim_{\mathscr{L}'_o} E_{\Lambda^c}^{\omega}(F) = \lim_{\mathscr{L}'_o} E_{\Lambda^c}^{\omega^o}(F) = \mu(F) \tag{27}$$

for μ-a.a., $\omega \in \Omega$.

Because $\mathscr{A}^{\uparrow}$ determines $\mathscr{M}$,

$$\lim_{\mathscr{L}'_o} E_{\Lambda^c}^{\omega} = \mu, \qquad \mu\text{-a.e.} \tag{28}$$

which is the necessary and sufficient condition for $\mu \in \partial G(\mathscr{E})$ (see, e.g., Refs. 2, and 9).

The proof of the global Markov property (GMP) is based on the following simple lemma, which is proven.[4]

Lemma 1. Let, for any $Q \subset \mathbb{L}$, $\Lambda \in \mathbb{L}$, and $\omega^o \in \Omega$

$$\mu_{Q,\omega^o,\Lambda}(\cdot) := \mu E_{(\Lambda \cap Q)^c}^{\omega_{Q^c} \times \omega^o_Q}(\cdot) \tag{29}$$

If for any $Q \subset \mathbb{L}$

$$\lim_{\mathscr{L}_o} \mu_{Q,\omega^o,\Lambda} = \mu \tag{30}$$

then

$$\mu \in \mathrm{GMP}$$

□

To continue the proof of GMP: For any $F \in \mathscr{A}^{\uparrow}_{\tilde{\Lambda}}$, $\tilde{\Lambda} \subset \Lambda \in \mathscr{L}$ we have

$$\mu_{Q,\omega^o,\Lambda} F \equiv \mu E^{\omega_{Q^c} \times \omega^o_Q}_{(\tilde{\Lambda} \cap Q)^c}(F) = \mu E^{\omega}_{\Lambda^c} E^{\omega'_{Q^c} \times \omega^o_Q}_{(\tilde{\Lambda} \cap Q)^c}(F) \tag{31}$$

(where explicitly indicated the variables of integration ω and ω').

From the attractivity and the compatibility condition of $\mathscr{E}$ we have

$$\begin{aligned} E^{\omega \wedge \omega^o}_{\Lambda^c}(F) &= E^{\omega \wedge \omega^o}_{\Lambda^c} E^{\omega'_{Q^c} \times (\omega \wedge \omega^o)_Q}_{(\tilde{\Lambda} \cap Q)^c}(F) \leqslant E^{\omega \wedge \omega^o}_{\Lambda^c} E^{\omega'_{Q^c} \times \omega^o_Q}_{(\tilde{\Lambda} \cap Q)^c}(F) \\ &\leqslant E^{\omega}_{\Lambda^c} E^{\omega'_{Q^c} \times \omega^o_Q}_{(\tilde{\Lambda} \cap Q)^c}(F) \leqslant E^{\omega \vee \omega^o}_{\Lambda^c}(F) \end{aligned} \tag{32}$$

Hence, using (23) and the fact that $\bigcup_{\mathscr{L}} \mathscr{A}^{\uparrow}_{\Lambda}$ determines, $\mathscr{M}$ we obtain

$$\lim_{\mathscr{L}_o} \mu_{Q,\omega^o,\Lambda} = \mu \tag{33}$$

for any $Q \subset \mathbb{L}$, which together with Lemma 1 gives

$$\mu \in \text{GMP} \qquad \square$$

Remark. The condition (23) is fulfilled for FKG-maximal Gibbs measures (see Appendix). Note, that under the conditions $\exists \omega^o \in \Omega$

(i) $\quad \mu = \lim_{\mathscr{L}_o} E^{\omega^o}_{\Lambda^c}$

(ii) $\quad \mu = \lim_{\mathscr{L}_o} \mu E^{\omega \vee \omega^o}_{\Lambda^c} \quad$ or $\quad \mu = \lim_{\mathscr{L}_o} \mu E^{\omega \vee \omega^o}_{\Lambda^c}$

one can prove extremality (see Appendix), but not GMP (without any additional condition on ω^o as, for example, in Ref. 12, Proposition 1). $\square$

Condition (23) reflects the idea of the existence of a ground configuration ω^o of the infinite system in the state of thermodynamical balance.[10] If we have a system in a pure phase at sufficiently low temperature, then all its typical configurations ω fluctuate around ω^o and can differ considerably from ω^o only in the regions (islands) of small diameters, so $\omega \wedge \omega_o$ and $\omega \vee \omega^o$ are also typical configurations (for almost all ω) for a system in the same pure phase. (In the high temperature region we have rapid decay of correlations and so weak dependence of the state of a subsystem in a finite volume from the outside configurations of the rest of the system. Condition (23) can also be fulfilled in this case.)

In practice, we construct a Gibbs measure as the limit of a sequence of the measures $E^{\omega^o}_{\Lambda^c}$ for some ground configuration ω^o. Moreover, it is frequently possible to prove the uniform in volume $\Lambda \in \mathscr{L}$ cluster property for measures $E^{\omega^o}_{\Lambda^c}$.

We use the above facts in order to give an example of application of the criterion formulated in Proposition 1.

2. Let $\mathbb{L}$ have a metric $d(\cdot, \cdot)$. We denote diam $\Lambda := \sup_{x, y \in \Lambda} d(x, y)$. Let $d\rho \in \mathscr{M}$ be a free measure defined by

$$d\rho = \bigotimes_{i \in \mathbb{L}} d\rho_i \tag{34}$$

where $d\rho_i$ are the probability measures on $(Y_i, \mathscr{Y}_i)$, $i \in \mathbb{L}$. For any $\Lambda \subset \mathbb{L}$ let

$$d\rho_\Lambda = \bigotimes_{i \in \Lambda} d\rho_i \tag{35}$$

be the corresponding probability measure on $\mathsf{X}_{i \in \Lambda} (Y_i, \mathscr{Y}_i)$.

Let $\phi := \{\phi_X\}_{X \in \mathscr{L}}$, where ϕ_X are Σ_X measurable real functions, be an interaction potential. We will assume that ϕ is of finite range, i.e.

$$\exists r_o,\ 1 \leqslant r_o < \infty \qquad \text{if} \quad \text{diam}\, X > r_o, \text{ then } \phi_X \equiv 0 \tag{36}$$

We also assume that $\phi_X \in \mathscr{A}_X$ and define

$$\|\phi\| := \sup_{i \in \mathbb{L}} \sum_{\substack{X \in \mathscr{L} \\ i \in X}} \|\phi_X\|_\infty \tag{37}$$

Let for $\Lambda \in \mathscr{L}$

$$U_\Lambda(\omega) := \sum_{\substack{X \in \mathscr{L} \\ X \cap \Lambda \neq \varnothing}} \phi_X(\omega) \tag{38}$$

We define the local specifications $\mathscr{E}_\beta = \{E_{\Lambda^c}\}_{\Lambda \in \mathscr{L}}$ by

$$E^{\tilde{\omega}}_{\Lambda^c}(\cdot) := \delta_{\tilde{\omega}} \mid_{\Sigma_{\Lambda^c}} \left[\frac{\int d\rho_\Lambda(e^{-\beta U_\Lambda} \cdot)}{\int d\rho_\Lambda(e^{-\beta U_\Lambda})} \right] \tag{39}$$

where $\beta \in \mathbb{R}^+$ and $\delta_{\tilde{\omega}} \mid_{\Sigma_{\Lambda^c}}$ is the restriction of point measures $\delta_{\tilde{\omega}}$ to Σ_{Λ^c}.

If we define the set of nearest neighbors $\partial\{i\}^c$ of $i \in \mathbb{L}$ by

$$\partial\{i\}^c := \{j \in \mathbb{L} : d(i, j) \leqslant r_o\} \tag{40}$$

then we have from definitions (2) and (15) that $\mathscr{E}_\beta$ given by (39) are Markov.

If ϕ is an attractive potential, i.e.

$$\forall \omega, \tilde{\omega} \in \Omega \quad \forall X \in \mathscr{L} \quad \phi_X(\omega \vee \tilde{\omega}) + \phi_X(\omega \wedge \tilde{\omega}) \leqslant \phi_X(\omega) + \phi_X(\tilde{\omega}) \tag{41}$$

then $\mathscr{E}_\beta$ is attractive.[9]

From now on we restrict ourselves to $\mathbb{L}=\mathbb{Z}^2$, $d(i,j)=(|i^1-j^1|+|i^2-j^2|)$. We assume that $\mathscr{L}_o$ consists of $\Lambda\in\mathscr{L}$ for which $|\partial\Lambda^c|=O(|\Lambda|^{1/2})=O[d(0,\partial\Lambda^c)]$ (where $|X|$ means the number of points in $X\in\mathscr{L}$). For $F\in\mathscr{A}_{\Lambda_1}$, $G\in\mathscr{A}_{\Lambda_2}$ we denote $d(F,G)=d(\Lambda_1,\Lambda_2)=\inf_{i\in\Lambda_1,j\in\Lambda_2}d(i,j)$. (For a given F, G we take here the smallest possible sets Λ_1 and Λ_2.) Let Δ_i be the cube with center at a point $i\in\mathbb{L}$ and a side of length $r=2s$, $s\in N$. For a finite union of such disjoint cubes

$$\Gamma=\bigcup\{\Delta_{i_k},k=1,\ldots,\imath\}\imath\in\mathbb{N}$$

we write $i\in\Gamma$ if and only if $\Delta_i\subset\Gamma$ and $|\Gamma|$ will denote the number of cubes in Γ. Let $\mathbb{L}_r=[\bigcup\Delta_i\colon i=(rk_1,rk_2),k_1,k_2\in\mathbb{Z}\}$.

We may and do assume that any element of $\mathscr{L}_o$ is a subset of $\mathbb{L}_r$.

Let χ_i^o be a characteristic function of a subset: $A_i\subset\mathsf{X}_{j\in\Delta_i}Y_j$ and $\omega^o|_{\Delta_i}\in A_i$ for some $\omega^o\in\Omega$. We write $\chi_i\equiv 1-\chi_i^o$ and

$$\chi_\Gamma^o:=\prod_{i\in\Gamma}\chi_i^o,\qquad \chi_\Gamma:=\prod_{i\in\Gamma}\chi_i[\operatorname{dist}(i,j)\geqslant r;\,i,j\in\Gamma] \tag{42}$$

With the above notation we have

Proposition 2. Let $\mathscr{E}_\beta$ be local attractive and Markov specifications on (Ω,Σ) given by (38) with $\|\phi\|<\infty$ and range of ϕ equal to $r_o\leqslant r$. Let the following conditions be fulfilled

(i) Let $\mu\in G(\mathscr{E}_\beta)$ and for some $\omega^o\in\Omega$

$$\mu=\lim_{\mathscr{L}_o}E_{\Lambda^c}^{\omega^o}$$

(ii) There is $\beta_o>0$, that if $\beta>\beta_o$ then

$$\mu\chi_\Gamma\leqslant C_1e^{-2\alpha\beta|\Gamma|} \tag{43}$$

with C_1, $\alpha>0$ constants independent of β.

(iii) For $\omega\in\mathsf{X}_{i\in\mathbb{L}_r}A_i\equiv A$ and $\Lambda\in\mathscr{L}_o$, the measures $E_{\Lambda^c}^\omega$ have uniform (in volume and boundary conditions) cluster property, i.e., for Λ_1, $\Lambda_2\in\mathscr{L}$, with $d(\Lambda_1,\Lambda_2)$ sufficiently big and any $F\in\mathscr{A}_{\Lambda_1}$, $G\in\mathscr{A}_{\Lambda_2}$

$$|E_{\Lambda^c}^\omega(F,G)|\leqslant\|F\|_\infty\,\|G\|_\infty\,C_2e^{-m(\beta)\,d(F,G)} \tag{44}$$

where $C_2>0$ is a constant independent of β. Assume furthermore that

$$m(\beta)\xrightarrow[\beta\to\infty]{}\infty$$

Then we have

$$\mu \in \partial G(\mathscr{E}_\beta) \cap \mathrm{GMP}$$

Proof. Without restriction of generality we can and do assume that $\Lambda_n \in \mathscr{L}_o$ are the squares with center at the point $i=0$ and side of length $n \cdot r$, $r \geqslant r_o$ (r_o is the range of interaction). Then $\partial \Lambda_n^c$ can be covered by $4n+4$ disjoint cubes of side r. Let $F \in \mathscr{A}_{\tilde{\Lambda}}$, $\tilde{\Lambda} \in \mathscr{L}$. We have, for $\tilde{\Lambda} \subset \Lambda \in \mathscr{L}$

$$\begin{aligned}
&\mu\, |E_{\Lambda^c}^{\omega \vee \omega^o}(F) - E_{\Lambda^c}^{\omega^o}(F)| \\
&\quad = \sum_{\Gamma \subset \partial \Lambda^c} \mu \chi^o_{\partial \Lambda^c \setminus \Gamma} \chi_\Gamma\, |E_{\Lambda^c}^{\omega \vee \omega^o}(F) - E_{\Lambda^c}^{\omega^o}(F)| \\
&\quad = \Bigg(\sum_{\substack{\Gamma \subset \partial \Lambda^c \\ |\Gamma| < N(\Lambda)}} + \sum_{\substack{\Gamma \subset \partial \Lambda^c \\ |\Gamma| \geqslant N(\Lambda)}} \Bigg) \mu \chi^o_{\partial \Lambda^c \setminus \Gamma} \chi_\Gamma\, |E_{\Lambda^c}^{\omega \vee \omega^o}(F) - E_{\Lambda^c}^{\omega^o}(F)|
\end{aligned} \tag{45}$$

where the summation is going over all possible families Γ consisting of cubes, and $N(\Lambda)$ is a natural number dependent on Λ, which will be chosen suitably later on.

Let us first estimate the second sum on the right-hand side of (45). Using (ii) we have

$$\begin{aligned}
&\sum_{\substack{\Gamma \subset \partial \Lambda \\ |\Gamma| \geqslant N(\Lambda)}} \mu \chi^o_{\partial \Lambda^c \setminus \Gamma} \chi_\Gamma\, |E_{\Lambda^c}^{\omega \vee \omega^o}(F) - E_{\Lambda^c}^{\omega^o}(F)| \\
&\quad \leqslant 2\, \|F\|_\infty \sum_{\substack{\Gamma \subset \partial \Lambda^c \\ |\Gamma| \geqslant N(\Lambda)}} \mu \chi_\Gamma \leqslant 2\, \|F\|_\infty \sum_{K=N(\Lambda)}^{M(\Lambda)} \binom{M(\Lambda)}{K} C_1 (e^{-\alpha\beta})^K
\end{aligned} \tag{46}$$

where

$$M(\Lambda) \equiv \frac{1}{r^2}\, |\partial \Lambda^c| \tag{47}$$

Now, for any $q > 0$

$$\begin{aligned}
\sum_{K=N}^{M} \binom{M}{K} q^K &= q^N \binom{M}{N} \sum_{K=0}^{M-N} \binom{M-N}{K} q^K \frac{N!\, K!}{(N+K)!} \\
&\leqslant q^N \binom{M}{N} (1+q)^{M-N}
\end{aligned} \tag{48}$$

We want to choose $N = [M/a] + 1$ (where $[x]$ is the biggest integer less than or equal to x) for some $a > 1$. In this case, remembering that M will be arbitrarily large, we can use Stirling's formula and get the estimate

$$\binom{M}{\left[\frac{M}{a}\right] + 1} \leqslant (a \cdot e)^{M/a} \frac{a(a-1)}{M^{1/2}} \tag{49}$$

From that

$$q^{[M/a]+1}\left(\frac{M}{\left[\frac{M}{a}\right]+1}\right)(1+q)^{M-[M/a]-1}$$

$$\leqslant [q\cdot ae\cdot(1+q)^{a-1}]^{M/a}\frac{q(a-1)a}{M^{1/2}} \tag{50}$$

We see that if we choose $a>1$ such that

$$q\cdot a\cdot e(1+q)^{a-1}\leqslant 1 \tag{51}$$

then the sum (48) will go to zero as $M\uparrow\infty$.

Because $a<e^{a-1}$ for $a>1$, $qae(1+q)^{a-1}<q(1+q)^{-1}[e(1+q)]^a$, and (51) will be fulfilled if we take

$$a\leqslant\frac{\ln(1+q)+|\ln q|}{\ln(1+q)+1} \tag{52}$$

We will simply choose

$$a=\tfrac{1}{2}|\ln q| \tag{53}$$

because in our case $q=e^{-2\alpha\beta}$. Therefore

$$a=\alpha\beta \tag{54}$$

and

$$N(\Lambda)=\left[\frac{M(\Lambda)}{a}\right]+1=\left[\frac{|\partial\Lambda^c|}{r^2\alpha\beta}\right]+1 \tag{55}$$

The second sum on the right-hand side of (45) will go to zero as $\Lambda\uparrow\mathbb{L}$. Let us now estimate the first sum on the right-hand side of (45) with $N(\Lambda)$ given by (55).

Let, for

$$\Gamma\subset\partial\Lambda^c,\qquad \Gamma^*\subset\partial\Lambda \tag{56}$$

Γ^* consisting of disjoint cubes Δ with sides of length r such that

$$\forall\Delta\subset\Gamma^*,\qquad d(\Delta,\Gamma)\leqslant r \tag{57}$$

Moreover, let us for $\omega\in\Omega$ define $\omega^A\in A$ by

$$\omega^A|_{\Delta_i}=\begin{cases}\omega\vee\omega_o|_{\Delta_i} & \omega\vee\omega_o|_{\Delta_i}\in A_i\\ \omega_o|_{\Delta_i} & \text{otherwise}\end{cases} \tag{58}$$

We have

$$E^{\omega \vee \omega^o}_{\Lambda^c}(F) = \frac{E^{\omega^A}_{\Lambda^c}[e^{-\beta[U_\Lambda(\omega \vee \omega^o|_{\Lambda^c} x\cdot) - U_\Lambda(\omega^A|_{\Lambda^c} x\cdot)]}F(\cdot)]}{E^{\omega^A}_{\Lambda^c} e^{-\beta[U_\Lambda(\omega \vee \omega^o|_{\Lambda^c} x\cdot) - U_\Lambda(\omega^A|_{\Lambda^c} x\cdot)]}}$$

$$= E^{\omega^A}_{\Lambda^c}(F) + \frac{E^{\omega^A}_{\Lambda^c}[e^{-\beta[U_\Lambda(\omega \vee \omega^o|_{\Lambda^c} x\cdot) - U_\Lambda(\omega^A|_{\Lambda^c} x\cdot)]}, F(\cdot)]}{E^{\omega^A}_{\Lambda^c}(e^{-\beta[U_\Lambda(\omega \vee \omega^o|_{\Lambda^c} x\cdot) - U_\Lambda(\omega^A|_{\Lambda^c} x\cdot)]})} \tag{59}$$

Using (iii) we have

$$|E^{\omega^A}_{\Lambda^c}[e^{-\beta[U_\Lambda(\omega \vee \omega^o|_{\Lambda^c} x\cdot) - U_\Lambda(\omega^A|_{\Lambda^c} x\cdot)]}, F(\cdot)]|$$
$$\leqslant \sup_{\omega,\omega^A,\tilde\omega} (e^{-\beta[U_\Lambda(\omega \vee \omega^o|_{\Lambda^c} x\tilde\omega_\Lambda) - U_\Lambda(\omega^A|_{\Lambda^c} x\tilde\omega_\Lambda)]})$$
$$\times \|F\|_\infty C_2 e^{-m(\beta)\, d(\tilde\Lambda, \partial\Lambda)} \tag{60}$$

because the first function on the left-hand side of (60) is in $\mathscr{A}_{\partial\Lambda}$ and $F \in \mathscr{A}_{\tilde\Lambda}$. Because

$$\sup_{\omega,\omega^A,\tilde\omega} |U_\Lambda(\omega \vee \omega^o|_{\Lambda^c} \times \tilde\omega_\Lambda) - U_\Lambda(\omega^A|_{\Lambda^c} \times \tilde\omega_\Lambda)| \leqslant 2\,\|\phi\|\, 3r^2|\Gamma| \tag{61}$$

where Γ is the union of cubes $\Delta_i \subset \partial\Delta^c$ in which $\omega \vee \omega^o|_{\Delta_i} \notin A_i$, we have (using 59, 60, and 61)

$$|E^{\omega \vee \omega^o}_{\Lambda^c}(F) - E^{\omega^o}_{\Lambda^c}(F)| \leqslant |E^{\omega^A}_{\Lambda^c}(F) - E^{\omega^o}_{\Lambda^c}(F)| + e^{12\beta\,\|\phi\|\, r^2\,|\Gamma|}$$
$$\times \|F\|_\infty C_2 e^{-m(\beta)\, d(\tilde\Lambda, \partial\Lambda)} \tag{62}$$

Remembering that we have

$$|\Gamma| < \left[\frac{|\partial\Lambda^c|}{r^2\alpha\beta}\right] + 1 \qquad \text{i.e.} \qquad |\Gamma| \leqslant \frac{|\partial\Lambda^c|}{r^2\alpha\beta}$$

we can estimate the second term in (62) by

$$\|F\|_\infty C_2' e^{(48r^2/\alpha)\|\phi\|\, d(0,\partial\Lambda^c)} \times e^{-m(\beta)\, d(0,\partial\Lambda^c)} \tag{63}$$

with a constant $C_2' < \infty$ (independent of β). Because by assumption (iii)

$$m(\beta) \xrightarrow[\beta \to \infty]{} \infty$$

there exists β_o such that

$$\forall \beta > \beta_o \quad \frac{48r^2}{\alpha}\|\phi\| < m(\beta) \tag{64}$$

For such values of the inverse temperature β the sequence (63) goes to zero as $\Lambda \uparrow \mathbb{L}$ (through $\mathscr{L}_o$).

We now prove that the first term on the right-hand side of (62) goes to zero as $\Lambda \uparrow \mathbb{L}$. In fact, we can prove more, namely, that for any ω^Λ, $\tilde{\omega}^\Lambda \in A$ and $F \in \bigcup_{\Lambda' \in \mathscr{L}} \mathscr{A}_{\Lambda'}$

$$|E^{\omega^\Lambda}_{\Lambda^c}(F) - E^{\tilde{\omega}^\Lambda}_{\Lambda^c}(F)| \xrightarrow[\Lambda \uparrow \mathbb{L}]{} 0$$

Let $\{\omega^{\Lambda,\iota} \in A\}_{\iota=o}^{|\partial\Lambda^c \setminus \Gamma|/r^2}$ be the sequence such that $\omega^{\Lambda,0} = \omega^o$, $\omega^{\Lambda,|\partial\Lambda^c \setminus \Gamma|/r^2} = \omega^\Lambda$ and for any ι, $\omega^{\Lambda,\iota+1}$ differs from $\omega^{\Lambda,\iota}$ exactly on the one cube Δ_ι and $\omega^{\Lambda,\iota} \prec \omega^{\Lambda,\iota+1}$.

We have

$$|E^{\omega^\Lambda}_{\Lambda^c}(F) - E^{\omega^o}_{\Lambda^c}(F)| \leqslant \sum_{\iota=o}^{(|\partial\Lambda^c \setminus \Gamma|/r^2)-1} |E^{\omega^{\Lambda,\iota+1}}_{\Lambda^c}(F) - E^{\omega^{\Lambda,\iota}}_{\Lambda^c}(F)| \tag{65}$$

But

$$E^{\omega^{\Lambda,\iota+1}}_{\Lambda^c}(F) = E^{\omega^{\Lambda,\iota}}_{\Lambda^c}(F) + \frac{E^{\omega^{\Lambda,\iota}}_{\Lambda^c}[e^{-\beta[U_\Lambda(\omega^{\Lambda,\iota+1}|_{\Lambda^c}x\cdot) - U_\Lambda(\omega^{\Lambda,\iota}|_{\Lambda^c}x\cdot)]}, F(\cdot)]}{E^{\omega^{\Lambda,\iota}}_{\Lambda^c}(e^{-\beta[U_\Lambda(\omega^{\Lambda,\iota+1}|_{\Lambda^c}x\cdot) - U_\Lambda(\omega^{\Lambda,\iota}|_{\Lambda^c}x\cdot)]})} \tag{66}$$

and using (iii)

$$\frac{E^{\omega^{\Lambda,\iota}}_{\Lambda^c}[e^{-\beta[U_\Lambda(\omega^{\Lambda,\iota+1}|_{\Lambda^c}x\cdot) - U_\Lambda(\omega^{\Lambda,\iota}|_{\Lambda^c}x\cdot)]}, F(\cdot)]}{E^{\omega^{\Lambda,\iota}}_{\Lambda^c}(e^{-\beta[U_\Lambda(\omega^{\Lambda,\iota+1}|_{\Lambda^c}x\cdot) - U_\Lambda(\omega^{\Lambda,\iota}|_{\Lambda^c}x\cdot)]})}$$
$$\leqslant e^{12\beta\|\phi\|} \|F\|_\infty C_2 e^{-m(\beta)\, d(\tilde{\Lambda},\partial\Lambda)} \tag{67}$$

(where $F \in \mathscr{A}_{\tilde{\Lambda}}$, $\tilde{\Lambda} \in \mathscr{L}$).

From (66) and (67) we obtain

$$|E^{\omega^\Lambda}_{\Lambda^c}(F) - E^{\omega^o}_{\Lambda^c}(F)| \leqslant \|F\|_\infty C_3 |\partial\Lambda^c| e^{-m(\beta)\, d(0,\partial\Lambda^c)} \tag{68}$$

where C_3 is a constant independent of Λ.

If we go with $\Lambda \uparrow \mathbb{L}$ through $\mathscr{L}_o$ the left-hand side of (68) goes to zero. This ends the proof that

$$\lim_{\mathscr{L}_o} \mu E^{\omega \vee \omega^o}_{\Lambda^c} = \lim_{\mathscr{L}_o} E^{\omega^o}_{\Lambda^c} = \mu \tag{69}$$

because the set $\bigcup_{\Lambda \in \mathscr{L}} \mathscr{A}_\Lambda$ determines $\mathscr{M}$.

The second case

$$\lim_{\mathscr{L}_o} \mu E^{\omega \wedge \omega^o}_{\Lambda^c} = \lim_{\mathscr{L}_o} E^{\omega^o}_{\Lambda^c} = \mu \tag{70}$$

can be proved analogously, so from Proposition 1 we conclude that

$$\mu = \lim_{\mathscr{L}_o} E^{\omega^o}_{\Lambda^c} \in \partial G(\mathscr{E}) \cap \mathrm{GMP}$$

Remarks. We see that without the attractivity assumption on $\mathscr{E}$ the conditions (i)–(iii) imply $\mu \in \partial G(\mathscr{E})$. Also, the formulation of Proposition 2 admits various modifications. In particular, let us note that the fact that $\mathbb{L}$ is a lattice is completely nonessential. □

3. Let $Y_i = Y_o$, $i \in \mathbb{L} \equiv \mathbb{Z}^\nu$ be a finite set.

A configuration $\omega \in \Omega$ is called periodic if there exists $n \in \mathbb{N}$ such that $\forall x \in \mathbb{Z}^\nu_n \quad \omega_{i+x} = \omega_i$, $i \in \mathbb{L}$.

For configurations ω, $\tilde{\omega} \in \Omega$ we write $\omega = \tilde{\omega}$ (a.s.) if and only if $\omega_i = \tilde{\omega}_i$ for almost all $i \in \mathbb{L}$, i.e., beyond a finite set of the lattice points.

We define, for $\omega = \tilde{\omega}$ (a.s.) the relative Hamiltonian

$$H(\omega \mid \tilde{\omega}) := \lim_{\mathscr{L}_o} [U_\Lambda(\omega) - U_\Lambda(\tilde{\omega})] \tag{71}$$

where $U_\Lambda(\cdot)$ is defined in (38) in terms of a finite range potential ϕ. We assume that ϕ is periodic, i.e.

$$\forall x \in \mathbb{Z}^\nu_n \qquad \forall X \in \mathscr{L} \qquad \phi_{X+x}(\omega_{+x}) = \phi_X(\omega) \tag{72}$$

where $(\omega_{+x})_i = \omega_{i+x}$, $n \in \mathbb{N}$.

We say that a configuration $\omega^o \in \Omega$ is the ground configuration for the relative Hamiltonian $H(\cdot \mid \cdot)$ if and only if

$$\forall \omega \in \Omega \qquad \omega = \omega^o \text{ (a.s.)} \Rightarrow H(\omega \mid \omega^o) \geqslant 0 \tag{73}$$

Let $g(H)$ be the set of periodic ground configurations. Let us assume that $g(H)$ is a nonempty finite set. Let N_o be the common period for the Hamiltonian H (i.e., for ϕ) and for all elements of $g(H)$. Let $r > \max(N_o, r_o)$, where r_o is the range of ϕ, and let Δ^r_i be the r cube, i.e., the cube with center in $i \in \mathbb{L}$ and the side of length r. The boundary of configuration $\omega \in \Omega$ is the set

$$\partial\omega := \bigcup_{i \in \mathbb{L}} \{\Delta^r_i : \omega \mid_{\Delta^r_i} \neq \omega^q \mid_{\Delta^r_i}, \forall \omega^q \in g(H)\} \tag{74}$$

Peierl's condition. A Hamiltonian H fulfils Peierl's condition if and only if

$$\exists \varepsilon > 0 \qquad \forall \omega = \omega^q \text{ (a.s.) } H(\omega \mid \omega^q) \geqslant \varepsilon \, |\partial\omega| \tag{75}$$

where $\omega^q \in g(H)$.

From the works of Refs. [5, 3, 7, 8] (see also Ref. 10) we have

Proposition 3. Let $\mathscr{E}_\beta$ be given by (39) with a periodic potential ϕ for which the relative Hamiltonian H has a finite set $g(H)$ of periodic ground configurations. If H satisfies Peierl's condition (75) then there is a $\beta_o > 0$ such that for each $\beta > \beta_o$ the measures

$$\mu^q := \lim_{\mathscr{L}_o} E^{\omega^q}_{\Lambda^c}, \qquad \omega^q \in g(H) \tag{76}$$

exist. For these measures the condition (ii) of Proposition 2 is fulfilled with some constants C_1 and α, and with

$$\chi_i(\omega_i) = 1 - \delta_{\omega_i^q}(\omega_i) \tag{77}$$

(i.e., for measure μ^q we take the sets $A_j := \{\omega^q \mid_{\Delta_j'}\}$). Moreover, the measures $E^{\omega^q}_{\Lambda^c}$ have the uniform cluster property (44) with

$$m(\beta) \xrightarrow[\beta \to \infty]{} \infty \qquad \square$$

Remark. It is known from above references that $\mu^q \in \partial G(\mathscr{E}_\beta)$.

The above proposition gives us the following:

Corollary 1. Let $\mathscr{E}_\beta$ fulfil all the conditions of Proposition 3 on a two-dimensional lattice. If $\mathscr{E}_\beta$ are attractive and Markov, then there is $\beta_o > 0$ such that for each $\beta > \beta_o$

$$\mu^q \in \partial G(\mathscr{E}_\beta) \cap \text{GMP}$$

(where μ^q are given by 76). $\square$

The simplest example of a lattice model with attractive interaction which possesses more than two extremal Gibbs measures and so has a non-empty set of non-FKG-maximal Gibbs measures is Pott's model. In this model the single-spin state space Y_i ($i \in \mathbb{L}$) consists of $n \geq 3$ points and the interaction potential is given by

$$\phi_X := \begin{cases} -\delta_{\omega_i \omega_{i'}} & \text{if } \; |i - i'| = 1 \text{ and } X = \{i, i'\} \\ 0 & \text{otherwise} \end{cases}$$

It is known that at sufficiently low temperatures β^{-1} there exist in this model at least n extremal Gibbs measures.[10]

For other examples of lattice models with attractive interaction which possess many phases.[6]

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APPENDIX

The alternative version of Proposition 1.

Proposition A1. Let $\mathscr{E}$ be an attractive specification on a standard Borel space (Ω, Σ) with a partial (measurable, directed upward and downward) order $\preccurlyeq$.

Let for $\mu \in G(\mathscr{E})$ the following conditions hold: There exists $\omega^o \in \Omega$ such that

$$\begin{aligned}
&\text{(i)} \quad \mu = \lim_{\mathscr{L}_o} E_{\Lambda^c}^{\omega^o} \\
&\text{(ii)} \quad \lim_{\mathscr{L}_o} \mu E_{\Lambda^c}^{\omega \vee \omega^o} = \mu \qquad \text{or} \qquad \lim_{\mathscr{L}_o} \mu E_{\Lambda^c}^{\omega \wedge \omega^o} = \mu
\end{aligned} \tag{A1}$$

then

$$\mu \in \partial G(\mathscr{E})$$

and if $\mathscr{E}$ is Markov and both conditions in (ii) hold,

$$\mu \in \text{GMP}$$

Proof. (Extremality) We consider only the first case in (ii), since for the second case the proofs are similar.

Let $F \in \mathscr{A}^{\uparrow}$, then

$$\begin{aligned}
\lim_{\mathscr{L}_o} \mu\, |E_{\Lambda^c}^{\omega \vee \omega^o} F - E_{\Lambda^c}^{\omega^o} F| &= \lim_{\mathscr{L}_o} \mu(E_{\Lambda^c}^{\omega \vee \omega^o} F - E_{\Lambda^c}^{\omega^o} F) \\
&= \lim_{\mathscr{L}_o} \mu E_{\Lambda^c}^{\omega \vee \omega^o} F - \lim_{\mathscr{L}_o} E_{\Lambda^c}^{\omega^o} F \\
&= 0
\end{aligned} \tag{A2}$$

Hence, since $\mathscr{A}^{\uparrow}$ is determining class

$$\lim_{\mathscr{L}'_o} E_{\Lambda^c}^{\omega \vee \omega^o} = \mu, \qquad \mu\text{-a.e.} \tag{A2}$$

for a subsequence $\mathscr{L}'_o \subset \mathscr{L}_o$.

But now for any $F \in \mathscr{A}^{\uparrow}$

$$\lim_{\mathscr{L}''_o} E^{\omega \vee \omega^o}_{\Lambda^c}(F) \geqslant \lim_{\mathscr{L}''_o} E^{\omega}_{\Lambda^c}(F) \tag{A3}$$

for a subsequence $\mathscr{L}''_o \subset \mathscr{L}'_o$ and μ-a.a., $\omega \in \Omega$ (the limits on the right-hand side of A3 exist from the martingale convergence theorem), so in particular for $F \in \mathscr{A}^{\uparrow}$, $F \geqslant 0$ using (ii) we obtain

$$\mu(F) \geqslant \lim_{\mathscr{L}''_o} E^{\omega}_{\Lambda^c}(F), \qquad \mu\text{-a.e.} \tag{A4}$$

From that we conclude

$$\mu(F) = \lim_{\mathscr{L}''_o} E^{\omega}_{\Lambda^c}(F), \qquad \mu\text{-a.e.} \tag{A5}$$

and since $\mathscr{A}^{\uparrow +}$ (the set of nonnegative elements in $\mathscr{A}^{\uparrow}$) is the determining class for $\mathscr{A}$, we have

$$\mu = \lim_{\mathscr{L}''_o} E^{\omega}_{\Lambda^c}, \qquad \mu\text{-a.e.} \tag{A6}$$

which means[2,9]

$$\mu \in \partial G(\mathscr{E}) \qquad \square$$

We now show that condition (23) can be fulfilled by FKG-maximal Gibbs measures for $\mathscr{E}$ (compact).

Proposition A2. Under the conditions of Proposition 1, if $\mathscr{E}$ is compact, then the FKG-maximal Gibbs measures $\mu_{\pm}$ fulfill the condition (23). □

Proof. First let us assume that Ω is compact and $\omega^{\pm}$ are its maximal elements with respect to the order $\preccurlyeq$. Then

$$\mu_{\pm} = \lim_{\mathscr{L}_o} E^{\omega^{\pm}}_{\Lambda^c} \tag{A7}$$

Let us consider only the case of the measure μ_+, since the second case is similar. Because we have

$$\omega \vee \omega^+ = \omega^+, \qquad \omega \wedge \omega^+ = \omega \tag{A8}$$

Therefore, from (A7) the condition (23) is fulfilled.

Let us now consider the general case when $\Omega \subset \mathbb{R}^{\mathbb{L}}$ has no maximal elements (with respect to $\preccurlyeq$). As previously, we also fix our attention on the case of μ_+ measure.

Let ω^+ be an element of Ω such that

$$\lim_{\mathscr{L}_o} \mu^+ \{\omega \in \Omega: \forall i \in \Lambda^c \ \omega_i \leqslant \omega_i^+\} = 1 \tag{A9}$$

(such element exists!). Since $\mathscr{E}$ is compact we have also

$$\mu^+ = \lim_{\mathscr{L}_o} E_{\Lambda^c}^{\omega^+} \tag{A10}$$

Denoting by χ_{Λ^c} the characteristic function of the set

$$\{\omega \in \Omega: \forall i \in \Lambda^c \ \omega_i < \omega_i^+\} \tag{A11}$$

we have for any $F \in \mathscr{A}$

$$\mu^+ E_{\Lambda^c}^{\omega \vee \omega^+}(F) = \mu^+ \chi_{\Lambda^c} E_{\Lambda^c}^{\omega^+}(F) + \mu^+(1 \setminus \chi_{\Lambda^c}) \, E_{\Lambda^c}^{\omega \vee \omega^+}(F) \tag{A12}$$

Hence, using (A9) and (A10), we have

$$\lim_{\mathscr{L}_o} \mu^+ E_{\Lambda^c}^{\omega \vee \omega^+} = \mu^+ \tag{A13}$$

On the other hand

$$\begin{aligned} \mu^+ E_{\Lambda^c}^{\omega \wedge \omega^+}(F) &= \mu^+ \chi_{\Lambda^c} E_{\Lambda^c}^{\omega}(F) + \mu^+(1 \setminus \chi_{\Lambda^c}) \, E_{\Lambda^c}^{\omega \wedge \omega^+}(F) \\ &= \mu^+(F) + \mu^+(1 \setminus \chi_{\Lambda^c})[E_{\Lambda^c}^{\omega \wedge \omega^+}(F) - E_{\Lambda^c}^{\omega}(F)] \end{aligned} \tag{A14}$$

Using (A9) we conclude

$$\lim_{\mathscr{L}_o} \mu^+ E_{\Lambda^c}^{\omega \wedge \omega^+} = \mu^+ \tag{A15}$$

(We have used that $\mathscr{A}$ is determining the class for $\mathscr{M}$.) □

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